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Cumulants of the QCD topological charge distribution

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ABSTRACT

The distribution of the QCD topological charge can be described by cumulants, with the lowest one being the topological susceptibility. The vacuum energy density in a θ-vacuum is the generating function for these cumulants. In this paper, we derive the vacuum energy density in SU(2) chiral perturbation theory up to next-to-leading order keeping different up and down quark masses, which can be used to calculate any cumulant of the topological charge distribution. We also give the expression for the case of SU(N) with degenerate quark masses. In this case, all cumulants depend on the same linear combination of low-energy constants and chiral logarithm, and thus there are sum rules between the N-flavor quark condensate and the cumulants free of next-to-leading order corrections.

1. Introduction

Because of the axial U(1) anomaly, there exists a θ-term in quantum chromodynamics (QCD) which is a topological term. The partition function of QCD in a θ-vacuum is given by

$$Z(\theta) = \int [DG][Dq][D\bar{q}]\, e^{-S_{\rm QCD}[G,q,\bar{q}]-i\theta Q}\,, \tag{1}$$

where $S_{\rm QCD}[G,q,\bar{q}]$ is the QCD action at $\theta=0$ with G and q being the gluon and quark fields, respectively, and Q is the topological charge

$$Q = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\int d^4x\, G^{\mu\nu}(x)G^{\rho\sigma}(x)\,, \tag{2}$$

with $G^{\mu\nu}(x)$ the gluon field strength tensor. In the Euclidean space with a finite space–time volume V, the partition function $Z(\theta)$ is dominated by the ground state, i.e. vacuum, energy of QCD for large enough V (see, e.g. Ref. [1]), and we have

$$Z(\theta) = e^{-V e_{\rm vac}(\theta)}, \qquad \text{or} \qquad e_{\rm vac}(\theta) = -\frac{1}{V}\ln Z(\theta)\,, \tag{3}$$

where $e_{\rm vac}(\theta)$ is the vacuum energy density in the θ-vacuum. The distribution of the topological charge can be described in terms of moments, which are the expectation values $\langle Q^{2n}\rangle_{\theta=0}$ with positive integer n, or cumulants defined as

$$c_{2n} = \left.\frac{d^{2n}e_{\rm vec}(\theta)}{d\theta^{2n}}\right|_{\theta=0}\,. \tag{4}$$

The leading cumulant is the topological susceptibility, $c_2=\chi_t$. It and the fourth cumulant are given by the well-known formulae

$$\chi_t = \frac{1}{V}\left\langle Q^2\right\rangle_{\theta=0} \qquad c_4 = -\frac{1}{V}\left(\left\langle Q^4\right\rangle - 3\left\langle Q^2\right\rangle^2\right)_{\theta=0}\,. \tag{5}$$

These topological quantities are important to understand the QCD vacuum as well as to extract physical observables from lattice simulations at a fixed topology [1,2]. They can be measured on lattice using various methods, see, e.g., Refs. [3–19].

For large volume and small quark masses, the strong interaction dynamics is determined by the Goldstone bosons originating from the spontaneous breaking of the light-quark chiral symmetry, and thus can be well described by chiral perturbation theory (CHPT) [20,21]. Both of χ_t and c_4 have been calculated in CHPT in both leading order (LO) and next-to-leading order (NLO) [22,23,1,24–28]. Earlier discussions in the large N_c limit can be found in Refs. [29,30]. The NLO calculations for χ_t in Refs. [25,28] and for c_4 in Ref. [28] were performed for an arbitrary number of flavors with different masses, and based on the generating functionals of CHPT [21] expanded around $\theta=0$ up to 2-point loops (up to 1-point tadpole loops for the topological susceptibility [25]).

In this paper, we will derive a general formula for the vacuum energy density in SU(2) chiral perturbation theory keeping different masses for the up and down quarks. The derivation involves a

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direct calculation of the logarithm of the determinant for the free Goldstone bosons in a θ-vacuum, and thus does not require an expansion up to a finite n-point loops. In this sense, it contains a summation of all one-loop diagrams at NLO in the chiral expansion, i.e. $\mathcal{O}(p^4)$ with p denoting a small momentum or Goldstone boson mass, contributing to the vacuum energy. The expression for the vacuum energy density can then be used to calculate *any* cumulant of the distribution of the QCD topological charge defined in Eq. (4).

It was emphasized in Ref. [28] that lattice simulations of these topological quantities with degenerate quarks are very interesting to pin down the N-flavor quark condensate. Although both the topological susceptibility and the fourth cumulant depend on several low-energy constants (LECs) in the NLO chiral Lagrangian, in addition to the quark condensate, the authors found an interesting linear combination, $\chi_t + N^2 c_4/4$ with N the number of flavors, independent of any LEC. Thus, such a combination is particularly suitable for extracting the N-flavor averaged quark condensate whose absolute value is

$$\Sigma_N = F_N^2 B_N\,, \tag{6}$$

where F_N, the pion decay constant, and B_N are defined in the chiral limit. For determinations of the quark condensate from lattice calculations of the topological susceptibility, we refer to Ref. [26,16,17]. Stimulated by this insight, we will also derive general expressions for the SU(N) vacuum energy density and cumulants with degenerate quarks. It turns out that all the cumulants depend on the same linear combination of the NLO LECs and chiral logarithm. As a result, one can construct linear combinations of the cumulants free of NLO corrections.

At this point, we notice that higher cumulants can be obtained from lower ones and moments using the following recursion relation

$$c_{2n} = (-1)^{n+1} \times \left[\frac{\langle Q^{2n}\rangle}{V} + \sum_{m=1}^{n-1}(-1)^m \binom{2n-1}{2m-1}\left\langle Q^{2(n-m)}\right\rangle c_{2m}\right]_{\theta=0}. \tag{7}$$

2. Vacuum energy in SU(2) chiral perturbation theory

2.1. Leading order

Because the θ-angle can be rotated to the phase of the quark mass matrix by an axial U(1) rotation, the θ-dependence of physical quantities can be studied by using a complex quark mass matrix. At LO, $\mathcal{O}(p^2)$, of SU(N) chiral perturbation theory, the vacuum energy density in a θ-vacuum for N quarks is given by

$$e_{\rm vac}^{(2)}(\theta) = -\frac{F_N^2}{4}\left\langle \chi_\theta U_0^\dagger + \chi_\theta^\dagger U_0\right\rangle, \tag{8}$$

where $\chi_\theta = 2B_N\mathcal{M}\exp(i\theta/N)$ with $\mathcal{M}$ being the real and diagonal quark mass matrix, and the vacuum alignment U_0 can be parametrized as a diagonal matrix $U_0 = \text{diag}\{e^{i\varphi_1}, e^{i\varphi_2}, \ldots, e^{i\varphi_N}\}$ with the constraint $\sum_i \varphi_i = 0$. The angles φ_i are determined by minimizing the vacuum energy. It is equivalent to removing the tree-level tadpole terms of the neutral Goldstone bosons which would induce vacuum instability [31–33].

In this section, we will study the case with $N = 2$. We will drop the subscripts in F_2 and B_2 to be consistent with the traditional notation in CHPT. With $U_0 = \text{diag}\{e^{i\varphi}, e^{-i\varphi}\}$, we have

$$e_{\rm vac}^{(2)}(\theta) = 2F^2 B\bar{m}\left(\cos\frac{\theta}{2}\cos\varphi - \epsilon\sin\frac{\theta}{2}\sin\varphi\right), \tag{9}$$

where $\bar{m} = (m_u + m_d)/2$ is the average mass of the up and down quarks and $\epsilon = (m_d - m_u)/(m_u + m_d)$ quantifies the strong isospin breaking. Minimizing the vacuum energy with respect to φ, one gets [1]

$$\tan\varphi = -\epsilon\tan\frac{\theta}{2}\,. \tag{10}$$

Substituting this into Eq. (9), we get the vacuum energy density at LO, up to an additive normalization constant [1]

$$e_{\rm vac}^{(2)}(\theta) = -F^2 \mathring{M}^2(\theta)\,, \tag{11}$$

where $\mathring{M}^2(\theta)$ is the LO pion mass squared in a θ-vacuum [1]

$$\mathring{M}^2(\theta) = 2B\bar{m}\cos\frac{\theta}{2}\sqrt{1+\epsilon^2\tan^2\frac{\theta}{2}}\,. \tag{12}$$

Notice that in the absence of the electromagnetic interaction, the neutral and charged pions have the same mass at LO. The cumulants of the distribution of the topological charge can then be easily obtained. For instance, the topological susceptibility and the fourth cumulant at LO are

$$\chi_t^{(2)} = \frac{1}{2}F^2 B\bar{m}\left(1-\epsilon^2\right),$$
$$c_4^{(2)} = -\frac{1}{8}F^2 B\bar{m}\left(1+2\epsilon^2-3\epsilon^4\right), \tag{13}$$

which have been derived before in Refs. [22,25,24].

2.2. Next-to-leading order

At NLO, there are contributions from both the tree-level terms in the $\mathcal{O}(p^4)$ chiral Lagrangian and one-loop diagrams. The vacuum energy density up to NLO is given by

$$e_{\rm vac}(\theta) = e_{\rm vac}^{(2)}(\theta) + e_{\rm vac}^{(4,\rm loop)}(\theta) + e_{\rm vac}^{(4,\rm tree)}(\theta)\,, \tag{14}$$

where $e_{\rm vac}^{(2)}(\theta)$ is given in Eq. (9), $e_{\rm vac}^{(4,\rm loop)}(\theta)$ is the one-loop contribution to be calculated later on, and the NLO tree-level contribution is

$$\begin{aligned} e_{\rm vac}^{(4,\rm tree)}(\theta) &= -\frac{l_3}{16}\left\langle \chi_\theta^\dagger U_0 + \chi_\theta U_0^\dagger\right\rangle^2 + \frac{l_7}{16}\left\langle \chi_\theta^\dagger U_0 - \chi_\theta U_0^\dagger\right\rangle^2 \\ &\quad - \frac{h_1+h_3}{4}\left\langle \chi_\theta^\dagger \chi_\theta\right\rangle - \frac{h_1-h_3}{2}\text{Re}\left(\det\chi_\theta\right) \\ &= -\mathring{M}^4(\theta)\left\{l_3 + l_7\left[\frac{(1-\epsilon^2)\tan(\theta/2)}{1+\epsilon^2\tan^2(\theta/2)}\right]^2\right\} \\ &\quad - 2B^2\bar{m}^2\left[(h_1+h_3)\left(1+\epsilon^2\right)\right. \\ &\quad \left. + (h_1-h_3)\left(1-\epsilon^2\right)\cos\theta\right], \end{aligned} \tag{15}$$

where l_3, l_7 and h_1, h_3 are the LECs and high-energy constants (HECs), respectively, in the NLO two-flavor chiral Lagrangian [20],[1] and we have used Eq. (10).[2] Because both l_3 and h_1 are ultraviolet

[1] Here we use the SU(2) × SU(2) notation rather than the O(4) one in the original paper, see, e.g., [34].

[2] In principle, the vacuum alignment determined by minimizing the LO vacuum energy gets shifted due to the presence of the higher order terms, l_7 in this case. However, this shift only provides a perturbation and is of one order higher compared to the angle φ in Eq. (10). It introduces CP-odd vertices (see, e.g., Refs. [35–37]) and does not affect CP-even quantities up to $\mathcal{O}(p^4)$, thus irrelevant for us. It is for this reason that the topological susceptibility up to NLO in the chiral expansion calculated in Ref. [28] agrees with that in Ref. [25], where the vacuum alignment was calculated by minimizing the LO and NLO vacuum energy, respectively.

(UV) divergent [20],

$$l_3 = l_3^r - \frac{\lambda}{2}, \qquad h_1 = h_1^r + 2\lambda, \tag{16}$$

with λ the divergence at the space–time dimension $d = 4$ in dimensional regularization,

$$\lambda = \frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} - \frac{1}{2}\left[\ln(4\pi) + \Gamma'(1) + 1\right]\right\}, \tag{17}$$

where μ is the scale in dimensional regularization, $e_{\rm vac}^{(4,{\rm tree})}(\theta)$ is UV divergent as well, and the divergence is (the divergence is nontrivial in a θ-vacuum noticing the θ-dependence)

$$e_{\rm vac}^{(4,{\rm tree},\infty)}(\theta) = -\frac{3\lambda}{2}\mathring{M}^4(\theta). \tag{18}$$

As will be shown, this divergence is exactly canceled by the one from loops in $e_{\rm vac}^{(4,{\rm loop})}(\theta)$.

Before proceeding to calculating the loop contribution to the vacuum energy density, let us discuss the main difference between our treatment (see below) and the one in Refs. [25,28]. In those papers the authors took the expression of the generational functional in Ref. [21]. It is normalized to the free fields at $\theta = 0$ (notice that Refs. [20,21] assume $\theta = 0$). Then the loops were calculated using the Goldstone boson masses at $\theta = 0$, and the θ-dependence is kept in the operator σ^χ defined as (we have replaced U containing quantum fluctuations of Goldstone bosons by U_0 relevant for the vacuum energy)

$$\sigma_{PQ}^\chi = \frac{1}{8}\left\langle\left\{\lambda_P, \lambda_Q^\dagger\right\}\left(\chi_\theta^\dagger U_0 + \chi_\theta U_0^\dagger\right)\right\rangle - \delta_{PQ}\mathring{M}_P^2(0), \tag{19}$$

where λ_P are linear combinations of the SU(N) generators introduced to diagonalize the LO mass term [21], and $\mathring{M}_P(0)$ are the LO Goldstone boson masses at $\theta = 0$. This amounts to an expansion around $\theta = 0$, which is perfectly fine for the calculation of the cumulants of the topological charge. However, we notice that the first term in the above equation is in fact $\delta_{PQ}\mathring{M}_P^2(\theta)$. If we expand the one-loop generating functional around the one for the free fields in a θ-vacuum,

$$Z_0(\theta) = \frac{i}{2}\ln\det D_0(\theta) = \frac{i}{2}{\rm Tr}\ln D_0(\theta), \tag{20}$$

where Tr stands for taking trace in both the flavor (this is the space of the adjoint representation which is 3-dimensional for the SU(2) case) and coordinate spaces, and $D_0(\theta)$ is a differential operator,

$$D_{0\,PQ}(\theta) = \delta_{PQ}\left[\partial_\mu\partial^\mu + \mathring{M}_P^2(\theta)\right], \tag{21}$$

then $\mathring{M}_P^2(0)$ in Eq. (19) needs to be replaced by $\mathring{M}_P^2(\theta)$ and σ_{PQ}^χ vanishes. As a result, the only term left in the one-loop generating functional relevant for the vacuum energy is $Z_0(\theta)$. Thus, the vacuum energy density is given by

$$e_{\rm vac}^{(4,{\rm loop})}(\theta) = -\frac{i}{2V}{\rm Tr}\ln D_0(\theta). \tag{22}$$

For the case of SU(2), because the neutral and charged pions have the same mass at LO, $\mathring{M}_P^2(\theta)$ is given by Eq. (12), and $D_0(\theta) = \mathbb{1}_{3\times3}[\partial_\mu\partial^\mu + \mathring{M}^2(\theta)]$, where the unit matrix has the dimension of the adjoint representation for SU(2). Extending these considerations to the case of N degenerate quark flavors and using dimensional regularization, we obtain

$$\begin{aligned} e_{\rm vac}^{(4,{\rm loop})}(\theta) &= -\frac{i}{2}(N^2-1)\int\frac{d^dp}{(2\pi)^d}\ln\left[-p^2 + \mathring{M}^2(\theta)\right] \\ &= \frac{i}{2}(N^2-1)\int\frac{d^dp}{(2\pi)^d}\int_0^\infty\frac{d\tau}{\tau}e^{-\tau[-p^2+\mathring{M}^2(\theta)]} \\ &= (N^2-1)\mathring{M}^4(\theta)\left\{\frac{\lambda}{2} - \frac{1}{128\pi^2}\left[1 - 2\ln\frac{\mathring{M}^2(\theta)}{\mu^2}\right]\right\}. \end{aligned} \tag{23}$$

For $N = 2$, one sees that the UV divergence cancels exactly the one in Eq. (18). The sum of Eqs. (9), (15) and (23) provides the vacuum energy density in a θ-vacuum up to NLO,

$$\begin{aligned} e_{\rm vac}(\theta) = -F^2\mathring{M}^2(\theta) - \mathring{M}^4(\theta)\Bigg\{&\frac{3}{128\pi^2}\left[1 - 2\ln\frac{\mathring{M}^2(\theta)}{\mu^2}\right] \\ &+ l_3^r + h_1^r - h_3 + l_7\left[\frac{(1-\epsilon^2)\tan(\theta/2)}{1+\epsilon^2\tan^2(\theta/2)}\right]^2\Bigg\}, \end{aligned} \tag{24}$$

where we have dropped θ-independent constant terms. The renormalized LEC l_3^r and HEC h_1^r are scale dependent [20] and this scale dependence cancels that in the chiral logarithm resulting in a scale-independent vacuum energy density in a θ-vacuum. This is the main result of our paper. It is then trivial to obtain the expression for any cumulant, and the lowest two are

$$\begin{aligned} \chi_t &= \frac{1}{2}F^2B\bar{m}\left(1-\epsilon^2\right)\Bigg\{1 - \frac{2B\bar{m}}{F^2}\Bigg(\frac{3}{32\pi^2}\ln\frac{2B\bar{m}}{\mu^2} \\ &\quad - 2\left[l_3^r + h_1^r - h_3 - l_7\left(1-\epsilon^2\right)\right]\Bigg)\Bigg\} + \mathcal{O}\left(p^6\right), \\ c_4 &= -\frac{1}{8}F^2B\bar{m}\left(1 + 2\epsilon^2 - 3\epsilon^4\right) \\ &\quad + B^2\bar{m}^2\left(1-\epsilon^2\right)\Bigg\{\frac{9}{128\pi^2}\left(1-\epsilon^2\right) + \frac{3}{32\pi^2}\ln\frac{2B\bar{m}}{\mu^2} \\ &\quad - 2\left[l_3^r + h_1^r - h_3 - l_7\left(1 + 2\epsilon^2 - 3\epsilon^4\right)\right]\Bigg\} + \mathcal{O}\left(p^6\right). \end{aligned} \tag{25}$$

They agree with the general N-flavor expressions in Ref. [28] for $N = 2$. Furthermore, in the isospin symmetric case, they depend on the same combination of the LECs and HECs, $l_3^r - l_7 + h_1^r - h_3$.

3. SU(N) with degenerate quark masses

The evaluation of the functional determinant $Z_0(\theta)$ or $e_{\rm vac}^{(4,{\rm loop})}(\theta)$ in Eq. (23) only requires the Goldstone bosons to be degenerate. Therefore, it is easy to generalize the result in the previous section to the case of SU(N) with degenerate quark masses.[3] The one-loop contribution to the vacuum energy density in a θ-vacuum is given by Eq. (23) as well with $\mathring{M}(\theta)$ replaced by the LO Goldstone boson mass for SU(N), see below.

When all the quarks are degenerate with a mass m, the vacuum is given by $U_0 = \mathbb{1}_{N\times N}$. With the $\mathcal{O}\left(p^4\right)$ Gasser–Leutwyler Lagrangian for SU(N) [21], we get the tree-level contribution, including both LO and NLO, to the vacuum energy density in a θ-vacuum

[3] For SU(N) with different quark masses, one may expand around $\theta = 0$, $\ln D_0(\theta) = \ln D_0(0) + D_0^{-1}(0)\,\Delta(\theta) + D_0^{-1}(0)\,\Delta(\theta)\,D_0^{-1}(0)\,\Delta(\theta) + \ldots$ with $\Delta_{PQ}(\theta) = \delta_{PQ}\left[\mathring{M}_P^2(\theta) - \mathring{M}_P^2(0)\right]$. This gives the general formulation used in Ref. [28].

$$e_{\mathrm{vac}}^{\mathrm{tree}} = -NF_N^2 B_N m \cos\frac{\theta}{N} - 4NB_N^2 m^2 \left(4NL_6\cos^2\frac{\theta}{N} - 4NL_7\sin^2\frac{\theta}{N} + 2L_8\cos\frac{2\theta}{N} + 4H_2\right), \tag{26}$$

where $L_{6,7,8}$ are LECs and H_2 is a HEC. Among them, L_6, L_7 and H_2 contain a UV divergent piece which can be calculated using the heat kernel method with path integral [21,38]

$$L_6 = L_6^r + \frac{N^2+2}{16N^2}\lambda, \qquad L_8 = L_8^r + \frac{N^2-4}{16N}\lambda,$$
$$H_2 = H_2^r + \frac{N^2-4}{8N}\lambda. \tag{27}$$

It is straightforward to check that these divergences cancel the one in $e_{\mathrm{vac}}^{(4,\mathrm{loop})}$ in Eq. (23). The vacuum energy density in a θ-vacuum up to NLO is then

$$e_{\mathrm{vac}}(\theta) = -\frac{N}{2}F_N^2\mathring{M}_N^2(\theta) - \mathring{M}_N^4(\theta)\left\{\frac{N^2-1}{128\pi^2}\left[1-2\ln\frac{\mathring{M}_N^2(\theta)}{\mu^2}\right] + 4N\left(NL_6^r + L_8^r - NL_7\tan^2\frac{\theta}{N}\right)\right\} \tag{28}$$

with the scale-dependent finite LECs L_6^r and L_8^r, where $\mathring{M}_N^2(\theta) = 2B_N m\cos(\theta/N)$, and the cumulants are

$$c_{2n} = \frac{(-1)^{n+1}}{N^{2n-1}}\left\{F^2B_Nm + 4^nB_N^2m^2\left[\frac{N^2-1}{64\pi^2N}\left(1-2\ln\frac{2B_Nm}{\mu^2}\right) + 8(N\,L_6^r + L_8^r + N\,L_7)\right]\right\} + \frac{N^2-1}{16\pi^2}B_N^2m^2\xi_{N,2n} \tag{29}$$

with the number $\xi_{N,2n}$ defined as

$$\xi_{N,2n} = \frac{d^{2n}}{d\theta^{2n}}\left[\cos^2\frac{\theta}{N}\,\ln\left(\cos\frac{\theta}{N}\right)\right]\bigg|_{\theta=0}. \tag{30}$$

One sees that all cumulants depend on the same linear combination of the LECs, as observed in Ref. [28] for the topological susceptibility and the fourth cumulant, and chiral logarithms. From this it is easy to construct LEC-free combination of cumulants which can be used for a clean extraction of the N-flavor quark condensate from lattice simulations as suggested in Ref. [28]. Examples are

$$\chi_t + \frac{N^2}{4}c_4 = \frac{3F_N^2B_Nm}{4N} + \frac{3\left(N^2-1\right)B_N^2m^2}{32\pi^2N^2} + \mathcal{O}\left(p^6\right),$$
$$\chi_t - \frac{N^4}{16}c_6 = \frac{15F_N^2B_Nm}{16N} + \frac{15\left(N^2-1\right)B_N^2m^2}{64\pi^2N^2} + \mathcal{O}\left(p^6\right), \tag{31}$$

where the first expression was already proposed in Ref. [28].[4] More interestingly, the NLO corrections can be canceled out completely in certain linear combinations, and lead to sum rules between the QCD topological sector and the spontaneous breaking of chiral symmetry, such as

$$\Sigma_N = \frac{N}{m}\left(\frac{8}{5}\chi_t + \frac{2N^2}{3}c_4 + \frac{N^4}{15}c_6\right) + \mathcal{O}\left(p^6\right). \tag{32}$$

In fact, in the chiral limit, we have the following exact relation as can be seen from Eq. (29)

$$\Sigma_N = \pi\,\rho(0) = \lim_{m\to 0}(-1)^{n+1}N^{2n-1}\frac{c_{2n}}{m}, \tag{33}$$

where we have displayed the Banks–Casher relation [39] linking the quark condensate to the zero-mode spectral density of the Euclidean Dirac operator, denoted by $\rho(0)$, as well. These relations can be simply obtained using the LO expression for the vacuum energy density, and suggest that there is an intimate link between the QCD topological sector and the spontaneous breaking of chiral symmetry.

4. Summary

We have derived the expressions for the vacuum energy density in a θ-vacuum in SU(2) CHPT up to NLO keeping different up and down quark masses as well as in SU(N) CHPT with degenerate quark masses. They can be used to calculate the cumulants of the QCD topological charge distribution which are important quantities to study QCD in the low-energy strong coupling regime. In the case of degenerate quark masses, all cumulants depend on the same linear combination of low-energy constants, as already observed for the topological susceptibility and the fourth cumulant in Ref. [28]. Therefore, one can construct many combinations of the cumulants depending only on the quark mass and condensate. They can be used to extract the quark condensate in lattice simulations without contamination from LECs. Furthermore, we find sum rules relating the quark condensate to the cumulants free of NLO corrections. It would be interesting to check such relations in lattice QCD.

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